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# Physicality of Weak Prandtl-Meyer Reflection*

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**Abstract**

Consider supersonic flow around a ramp. There are two possible shock reflections, the weak and strong Prandtl-Meyer reflection. Experimentally, it is the weak reflection that is observed. The purpose of this article is to present recent analytical formulations and computational results on this problem. Through numerical computations, we show that the usual time-asymptotic stability criterion fails to rule out the strong reflection. Instead, we consider the self-similar flows, the time-asymptotic states of general flows.

## 1 Introduction

Consider the isentropic Euler equations of compressible gas dynamics in two space dimensions:

$$\rho_t + \nabla \cdot (\rho\vec{v}) = 0 \tag{1}$$

$$(\rho\vec{v})_t + \nabla \cdot (\rho\vec{v} \otimes \vec{v}) + \nabla(p(\rho)) = 0, \tag{2}$$

here $\rho$ is the density, $\vec{v} = (u, v)$ the velocity, $\vec{x} = (x^1, x^2) = (x, y)$ the space variables, and $\nabla = \partial_{\vec{x}}$ the spatial gradients. We consider the polytropic gases so that, after some rescaling, the pressure function $p(\rho)$ is given by

$$p(\rho) = \rho^\gamma, \ \gamma \in (1, \infty).$$

The Euler equations possess two types of discontinuity, the shock waves as well as the highly unstable vortex sheets. To focus on the shock waves, we will

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consider the potential flow equations with the additional hypothesis that the flow is irrotational. That is, there exists a scalar potential function $\phi$ such that

$$\vec{v} = \nabla\phi.$$

From this and the Euler equations, (1), (2), the following Bernoulli law holds for smooth flows:

$$\begin{aligned} 0 &= \phi_{x_i t} + \nabla\phi_{x_i} \cdot \nabla\phi + \pi(\rho)_{x_i} \\ &= (\phi_t + \frac{|\nabla\phi|^2}{2} + \pi(\rho))_{x_i}, \end{aligned}$$

where $\pi(\rho)$ is related to the sound speed $c(\rho)$ by

$$\pi_\rho = c^2(\rho)/\rho, \;\; c^2(\rho) = p_\rho(\rho).$$

Thus, for some constant $A$,

$$\rho = \pi^{-1}(A - \phi_t - \frac{|\nabla\phi|^2}{2}).$$

Substituting this (1) yields the potential flow equation:

$$\phi_{tt} + 2\phi_x\phi_{tx} + 2\phi_y\phi_{ty} + [(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_x)^2 - c^2]\phi_{yy} = 0. \tag{3}$$

The stationary potential flow equation

$$[(\phi_x)^2 - c^2)]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_x)^2 - c^2)]\phi_{yy} = 0 \tag{4}$$

is elliptic when the flow is subsonic, $|\nabla\phi| < c$, and hyperbolic when it is supersonic $|\nabla\phi| > c$.

The problem we are interested in is the flow past a wedge of solid. The gas flow is in the region $\Omega$ outside of the wedge with angle $2\alpha$, in the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, $|\theta| \leq \pi$,:

$$\Omega = \{(x, y) : |\theta| > \alpha\}.$$

For constant supersonic upstream flow $\nabla\phi(x, y) = (\bar{u}, 0)$, $u > c$, the compression that the wedge induces gives raise to shocks. Using the shock polar analysis, Figure 1, [Courant-Friedrichs], Prandtl shows that there are two possible configurations, the weak shock reflection with supersonic downstream flow, Figure 2, and the strong shock reflection with subsonic downstream flow, Figure 3. The weak reflection is the one observed experimentally. Our main purpose is to address this analytically.

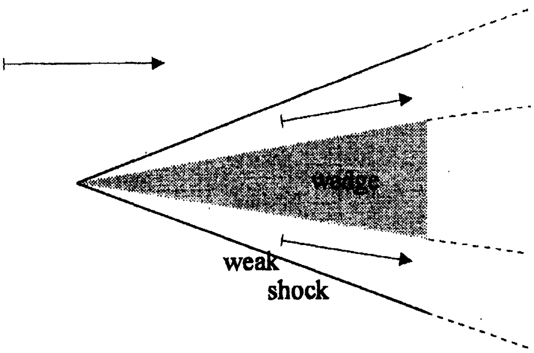


Figure 1: Weak shock

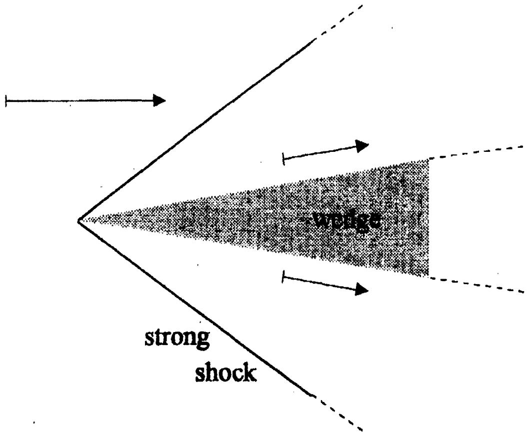


Figure 2: Strong shock

One plausible approach is to rule out the strong shock reflection by the consideration of stability, the stable ones are physical. Both weak and strong reflections are structural stable in that they exist upon small perturbation of the wedge. The time-asymptotic stability is the relevant one here. In Section 2 we present computational results on the time-asymptotic stability on both réfelctions. A surprising finding is that both strong and weak reflections are time-asymptotic stable upon compactly supported perturbations. Thus the consideration of time-asymptotic stability fails to rule out the strong reflection. The main part of our effort is to come up with an analytical setup of showing that the weak reflection is the one observed experimentally in that it represents the time-asymptotic state of an accelerating flow. This is explained in Section 3.

# 2 Computational Results on Time-Asymptotic Stability

We study the perturbation of the weak reflection, Figure 1, and of the strong reflection, Figure 2. The perturbation is compactly supported. We present results for the familiar Godunov method. More sophisticated numerical methods, as well as changes to upstream Mach number and other parameters, produce qualitatively similar results. The computational results are presented in Figure 3 for weak reflection and in Figure 4 for strong reflection. For weak reflection, the perturbation moves away from the tip of the wedge. This is because the downstream flow is supersonic. Due to dispersion and the absorbing nature of the leading shock, the perturbation decays in time. The downstream flow for strong reflection is subsonic and so the perturbation propagates both away from the tip of the wedge as well as toward the tip, and it also decays in time. In summary, both weak and strong reflections are nonlinear, time-asymptotically stable upon compactly supported perturbation. Thus this analysis fails to ex-

Figure 3: Weak shock perturbation (density shown)

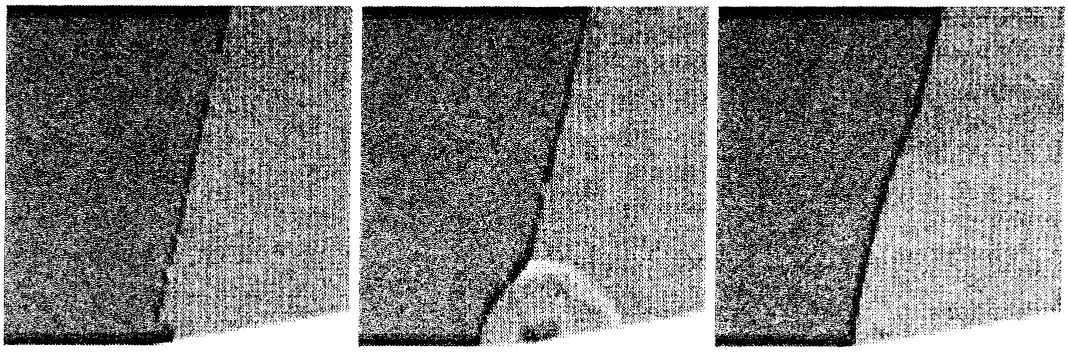

Figure 4: Strong shock perturbation (density shown)

plain the fact that experimentally it is the weak, not strong, reflection that is produced. We do, however, observe the instability of the strong reflection when the downstream state at $x = \infty$ is also perturbed; see Figure 5.

## 3 Self-Similar Flows

To produce the shock reflection, it is to accelerate the wedge to supersonic speed; or, equivalently, keep the wedge fixed and accelerate the upstream flow till the given constant supersonic velocity $(\bar{u}, o)$. The acceleration yields complex flow patterns and many shock waves are generated in the process. What interested us is what type of shock reflections will eventually be produced. By the time-asymptotic scaling $(x, y, t) \rightarrow s(x, y, t),\ s \rightarrow 0_+$, the process becomes the initial value problem with initial values at time $t = 0$:

$$\nabla\phi(x, y, 0) = (\bar{u}, 0).$$

In other words, the wedge is instantaneously accelerated to the supersonic state $(\bar{u}, 0)$. This simplifies greatly the thinking because both the geometry of the domain $\Omega$, the initial values, and the potential flow equation are invariant under

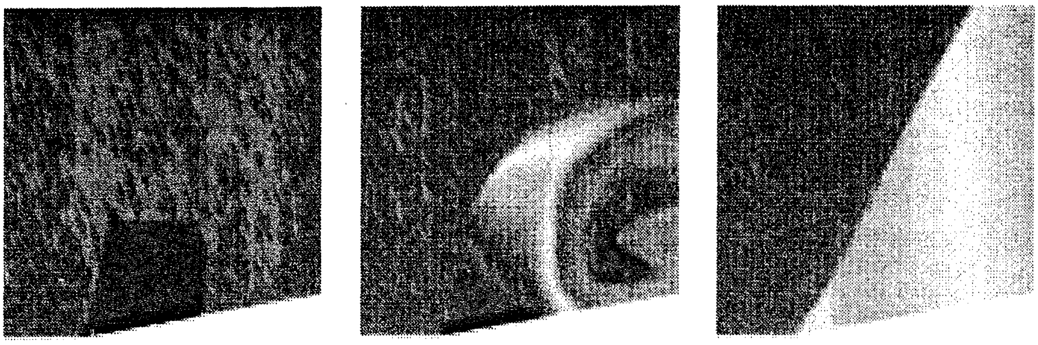

Figure 5: Strong shock vanishes under downstream state perturbation; right: the weak shock appears.

the scaling $(x, y, t) \to s(x, y, t)$. Thus the solution is self similar:

$$\phi(x, y, t) = t\psi(\xi, \eta), \ \xi = x/t, \ \eta = y/t.$$

The self-similar potential flow equation is

$$[c^2 - (\psi_\xi - \xi)^2]\psi_{\xi\xi} - 2(\psi_\xi - \xi)(\psi_\eta - \eta)\psi_{\xi\eta} + [c^2 - (\psi_\eta - \eta)^2]\psi_{\eta\eta} = 0. \quad (5)$$

The velocity takes similar form as before:

$$\nabla\psi = \vec{v}.$$

The self-similar potential flow equation is of mixed type. It is hyperbolic if the flow is pseudo-supersonic:

$$|\vec{v} - (\xi, \eta)| > c;$$

and elliptic if it is pseudo-subsonic:

$$|\vec{v} - (\xi, \eta)| > c.$$

The initial condition for the potential flow equation turns to the boundary condition for the self-similar equation at $|(\xi, \eta)| = \infty$. The self-similar solution represents the solution of the time-dependent solution at time $t = 1$. Since the flow is clearly pseudo-supersonic around $|(\xi, \eta)| = \infty$, the boundary condition consists of one-dimensional shock waves parallel to the ramp. We have carried out numerical computations. Our result shows that the self-similar solution consists of the weak shocks from the tip of the wedge, the one-dimensional shocks paralell to the ramp, curved shocks connecting the these two set of shocks, and a region of pseudo-subsonic flow bounded by the curved shock and two pseudo-sonic circles determined by the two constant states between the weak shocks, curved shocks and the ramp, Figure 6. We are finishing the analytical justification of this flow pattern. This would show that the weak shock reflection is the physical one.

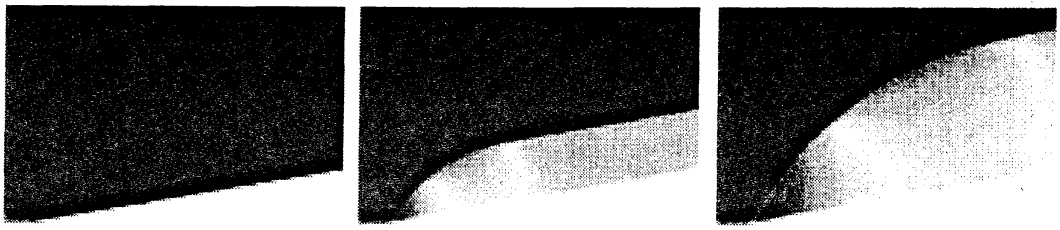

Figure 6: The weak shock appears spontaneously at the corner. It is connected to the straight reflected shock by a curved shock, with a nontrivial elliptic region below.

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